Faddeev-Jackiw quantization of an Abelian and non-Abelian exotic action for gravity in three dimensions

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A detailed Faddeev-Jackiw quantization of an Abelian and non-Abelian exotic action for gravity in three dimensions is performed. We obtain for the theories under study the constraints, the gauge transformations, the generalized Faddeev-Jackiw brackets and we perform the counting of physical degrees of freedom. In addition, we compare our results with those found in the literature where the canonical analysis is developed, in particular, we show that both the generalized Faddeev-Jackiw brackets and Dirac's brackets coincide to each other. Finally we discuss some remarks and prospects.

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I. INTRODUCTION

Nowadays, the study of singular systems becomes to be an important research subject. In fact, the gravitational interaction described by means Palatini's action, the physics of the fundamental interactions based on the standard model, Maxwell theory, Yang-Mills theories and BF theories [1–3] etc., are several examples of important singular systems, and its study at classical and quantum level becomes to be mandatory. In fact, the study of those dynamical systems has been developed through their symmetries and these symmetries form part of a relevant information in both the classical and quantum context. In this respect, it is well-known that there is a powerful formalism for studying the symmetries of singular systems, in particular those singular systems having an important symmetry called gauge symmetry, that formalism is know as the Dirac-Bergman method for constrained systems [4]. Dirac's canonical formalism is an elegant approach for obtaining relevant physical information of a theory, namely, the identification of the physical degrees of freedom, the gauge transformations, the complete structure of the constraints and the obtention of the extended action, all this information is useful because a strict study of the symmetries will allow us to have a guideline to make the best progress in the quantization. However, if a pure Dirac's canonical analysis is performed, in general it is complicated to develop the classification of the constraints in first and second class [5–7]; the classification of the constraints is an important step to perform because first class constraints are generators of gauge transformations and allow us

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identify observables. Moreover, second class constraints allow us to construct the Dirac brackets and also they are useful for identifying the Lagrange multipliers. Therefore, to develop all the steps of a pure canonical analysis becomes to be mandatory but it is not easy to perform [8].

On the other hand, there is an alternative method for studying singular systems in a different way, that is the well-known Faddeev-Jackiw [FJ] formalism [9]. The [FJ] framework is a symplectic description of constrained quantization, where the degrees of freedom are identified by means the so-called symplectic variables. In fact, for studying any theory we can choose as symplectic variables either configuration space or the phase space; in [FJ] framework there is a freedom for choosing the symplectic variables. Furthermore, as the system under study is singular there will be constraints and the [FJ] approach has the advantage that all the constraints of the theory are at the same footing, namely, it is not necessary perform the classification of the constraints in primary, secondary, first class or second class such as in Dirac's method is done. Moreover, in [FJ] approach also it is possible to obtain the gauge transformations of the theory and the generalized [FJ] brackets coincide with the Dirac's ones. The [FJ] framework has been applied to systems such as [YM] theory [10], QCD theory [11], first order Wess-Zumino terms [12], theories with extra dimensions [13] and several others interesting systems [14]. We can observe in all those works that the [FJ] approach is an elegant alternative for analysing gauge systems with certain advantages lacking the Dirac approach.

Because of the explained above, in this paper we perform both the Dirac and [FJ] analysis to an exotic Abelian theory and non-Abelian exotic action describing gravity in three dimensions. In fact, for the former we will show that if in Dirac's formalism we perform the analysis without fixing the gauge and we eliminate only the second class constraints through Dirac's brackets and remaining the first class ones, then it is possible reproduce those Dirac's results by working in [FJ] with the configuration space as symplectic variables, and using the temporal gauge in order to invert the symplectic matrix. Moreover, if in Dirac's method we perform the analysis and we fixing the gauge converting the first class constraints in second class and we construct the corresponding Dirac's brackets, then in [FJ] framework it is possible to reproduce these Dirac's results, but now we will work by using the phase space as symplectic variables; in order to invert the symplectic matrix we will fix the gauge using the Coulomb gauge. In addition, we will show that if in Dirac's method fixing or not the gauge, the constructed Dirac's brackets and the generalized [FJ] brackets coincide to each other. Furthermore, we will extend our analysis for a non-Abelian exotic theory describing gravity and we will reproduce in an elegant way by means [FJ] the results reported in [17] where the Dirac approach was performed, all these important results will be clarified along the paper.

The paper is organized as follows, in Sect. II we will perform the [FJ] analysis for an Abelian exotic action in three dimensions, we will obtain the constraints of the theory, the gauge transformations and we will carry out the counting of physical degrees of freedom concluding that the theory under study is a topological one. In addition, we will reproduce the results obtained from the canonical analysis where the second class constraints are eliminated by introducing the Dirac brackets and we show that the generalized [FJ] brackets coincide with those Dirac's brackets. Then, in Dirac's

method we will fix the gauge converting the first class constraints in second class and again, the Dirac brackets will be constructed, thus, in order to reproduce these results by using [FJ], we will perform our analysis by working now with the phase space as symplectic variables and we will show the equivalence between the generalized [FJ] and Dirac's brackets. In Sec. III we will extend our analysis developed in previous sections by performing the [FJ] analysis for the exotic action describing gravity, in particular, we will reproduce all the Dirac results reported in [17] where a pure canonical analysis was reported. In Sec. IV we provide a summary and prospects.

II. FADDEEV-JACKIW QUANTIZATION OF AN ABELIAN EXOTIC ACTION

It is well-known that in three dimensions there is an alternative action to Palatini's Lagrangian reproducing Einstein's equations with cosmological constant, that action is called exotic action for gravity. Exotic action for gravity can be seen as a limit of other theories such as topological gravity with torsion or topologically massive gravity [15–19], the action is given by

$$\begin{aligned} S^{Exotic}[A,e] \;&=\; \int_M A^I \wedge dA_I + \frac{1}{3}\epsilon_{IJK} A^I \wedge A^J \wedge A^K \\ &+\Lambda \int_M e^I \wedge d_A e_J. \end{aligned} \tag{1}$$

here, Λ is the cosmological constant, $A^I = A^I_\mu dx^\mu$ is the one-form in the adjoint representation of the Lie algebra of $SO(2,1)$ [18, 19], e^I corresponds to the dreibein field, $\mu, \nu = 0, 1, 2$ are spacetime indices, x^μ are the coordinates that label the points for the 3-dimensional spacetime manifold M and $I, J = 0, 1, 2$ are internal indices that can be raised and lowered by the internal metric $\eta_{IJ} = diag(-1, 1, 1)$. From (1) we can identify the Lagrangian density of an abelian exotic theory, this is

$$\mathcal{L} = \frac{1}{4}\epsilon^{\mu\nu\lambda}\left(A^I_\mu F_{I\nu\lambda} + \Lambda e^I_\mu(\partial_\nu e_{I\lambda} - \partial_\lambda e_{I\nu})\right), \tag{2}$$

where, A^I_μ is a set of three $U(1)$ gauge potentials, e^I_μ is the "frame" field and $F^I_{\nu\lambda} = \partial_\nu A^I_\lambda - \partial_\lambda A^I_\nu$ is the strength field. In this manner, we will develop the [FJ] analysis to the action (2). For this aim, we perform the 2+1 decomposition and we identify the first order symplectic Lagrangian given by

$$\mathcal{L}^{(0)} = \frac{1}{2}\epsilon^{0ij}(A^I_j \dot{A}_{iI} + \Lambda e^I_j \dot{e}_{iI}) - V^{(0)}, \tag{3}$$

where $V^{(0)} = -\frac{1}{2}\epsilon^{0ij}(A^I_0 F_{ijI} + 2\Lambda e^I_0 \partial_i e_{jI})$. The corresponding symplectic equations of motion are given by [9]

$$f^{(0)}_{ij}\dot{\xi}^j = \frac{\partial V^{(0)}(\xi)}{\partial \xi^i}, \tag{4}$$

where the symplectic matrix $f^{(0)}_{ij}$ takes the form

$$f^{(0)}_{ij}(x,y) = \frac{\delta a_j(y)}{\delta \xi^i(x)} - \frac{\delta a_i(x)}{\delta \xi^j(y)}, \tag{5}$$

with $\xi^{(0)i}$ and $a^{(0)}{}_i$ representing a set of symplectic variables. It is important to comment, that in [FJ] framework we are free for choosing the symplectic variables; we can choose either the configuration variables or the phase space variables. In fact, in order to obtain by means a different way the different scenarios applying the Dirac method, in this paper we will work with both. In this respect, let us reproduce by means the [FJ] method the results found in the Appendix A (see the subsection A), where a pure canonical analysis was developed for the action (1) and we have found the Dirac brackets by eliminating only the second class constraints and remaining the first class ones. So, for this aim we observe from the symplectic Lagrangian (3) that it is possible identify the following symplectic variables $\xi^{(0)i}(x) = \{e^I_i, e^I_0, A^I_i, A^I_0\}$ and the components of the symplectic 1-forms are $a^{(0)}{}_i(x) = \{\frac{1}{2}\Lambda\epsilon^{0ij}e_{Ij}, 0, \frac{1}{2}\epsilon^{0ij}A_{Ij}, 0\}$. Hence, by using our set of symplectic variables, the symplectic matrix (5) takes the form

$$
f^{(0)}_{ij}(x,y) = \begin{pmatrix} -\Lambda\epsilon^{0ij}\eta_{IJ} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon^{0ij}\eta_{IJ} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y), \tag{6}
$$

we observe that this matrix is singular. In fact, in [FJ] method this means that there are present constraints [20, 21]. In order to obtain these constraints, we calculate the zero modes of the symplectic matrix, the modes are given by $(v^{(0)}_i)^T_1 = (0, v^{e^I_0}, 0, 0)$ and $(v^{(0)}_i)^T_2 = (0, 0, 0, v^{A^I_0})$, where $v^{e^I_0}$ and $v^{A^I_0}$ are arbitrary functions. In this manner, by using the zero-modes and the symplectic potential $V^{(0)}$ we can get the following constraints

$$
\begin{aligned}
\Omega^{(0)}_I &= \int d^2x (v^{(0)})^T_i(x) \frac{\delta}{\delta\xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\
&= \int d^2x v^{e^I_0}(x)[-\Lambda\epsilon^{0ij}\eta_{IJ}\partial_i e_j{}^J] \rightarrow [-\Lambda\epsilon^{0ij}\eta_{IJ}\partial_i e_j{}^J] = 0,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\beta^{(0)}_I &= \int d^2x (v^{(0)})^T_i(x) \frac{\delta}{\delta\xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\
&= \int d^2x v^{A_0}(x)[-\frac{1}{2}\epsilon^{0ij}\eta_{IJ}\left[F^J{}_{ij}\right] \rightarrow [-\frac{1}{2}\epsilon^{0ij}\eta_{IJ}\left[F^J{}_{ij}\right] = 0.
\end{aligned} \tag{8}
$$

Now, we will observe if there are present more constraints in the context of [FJ]. For this aim, we write in matrix form the following system [20]

$$
f^{(1)}_{kj}\dot{\xi}^j = Z_k(\xi), \tag{9}
$$

where

$$
Z_k(\xi) = \begin{pmatrix} \frac{\partial V^{(0)}(\xi)}{\partial \xi^i} \\ 0 \\ 0 \end{pmatrix}, \tag{10}
$$

and

$$f_{kj}^{(1)} = \begin{pmatrix} f_{ij}^{(0)} \\ \frac{\partial \Omega^{(0)}}{\partial \xi^i} \\ \frac{\partial \beta^{(0)}}{\partial \xi^i} \end{pmatrix} = \begin{pmatrix} -\Lambda \epsilon^{0ij}\eta_{IJ} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon^{0ij}\eta_{IJ} & 0 \\ 0 & 0 & 0 & 0 \\ \Lambda \epsilon^{0ij}\eta_{IJ}\partial_i & 0 & 0 & 0 \\ 0 & 0 & \epsilon^{0ij}\eta_{IJ}\partial_i & 0 \end{pmatrix} \delta^2(x-y). \tag{11}$$

We can observe that the matrix (11) is not a square matrix as expected, however, it has linearly independent modes given by $(v^{(1)})_1^T = (\partial_i v^{\lambda}, v^{e_0^I}, 0, 0, v^{\lambda}, 0)$ and $(v^{(1)})_2^T = (0, 0, \partial_i v^{\alpha}, v^{A_0^I}, 0, v^{\alpha})$. These modes are used in order to obtain more constraints. In fact, by calculating the following contraction [20]

$$(v^{(1)})_k^T Z_k = 0, \tag{12}$$

where $k = 1, 2$, we obtain that (12) is an identity; thus, in the [FJ] context there are not more constraints for the theory under study.

Now, we will construct a new symplectic Lagrangian containing the information of the constraints obtained in (7) and (8). In order to archive this aim, we introduce to $e_0^I = \dot{\lambda}^I$ and $A_0^I = \dot{\theta}^I$ as Lagrange multipliers associated to those constraints, thus, we obtain the following symplectic Lagrangian

$$\mathcal{L}^{(1)} = \frac{1}{2}\epsilon^{0ij} A_{jI}\dot{A}_i^I + \frac{\Lambda}{2}\epsilon^{0ij} e_{jI}\dot{e}_i^I + (\Lambda \epsilon^{0ij}\partial_i e_{jI})\dot{\lambda}^I + (\epsilon^{0ij}\partial_i A_{jI})\dot{\theta}^I - V^{(1)}, \tag{13}$$

where $V^{(1)} = V^{(0)} \mid_{\Omega_I^{(0)}=0, \beta_I^{(0)}=0} = 0$, the symplectic potential vanish reflecting the general covariance of the theory just like it is present in General Relativity. In this manner, from (13) we identify the following new symplectic variables $\xi^{(1)i}(x) = \{e_i^I, \lambda^I, A_i^I, \theta^I\}$ and the new symplectic 1-forms $a^{(1)}{}_i(x) = \{\frac{1}{2}\Lambda \epsilon^{0ij} e_{Ij}, \Lambda \epsilon^{0ij}\partial_i e_{jI}, \frac{1}{2}\epsilon^{0ij} A_{Ij}, \epsilon^{0ij}\partial_i A_{jI}\}$. Hence, by using the new symplectic variables and 1-forms, we can calculate the following symplectic matrix

$$f_{ij}^{(1)}(x,y) = \begin{pmatrix} -\Lambda \epsilon^{0ij}\eta_{IJ} & -\Lambda \epsilon^{0ij}\eta_{IJ}\partial_j & 0 & 0 \\ \Lambda \epsilon^{0ji}\eta_{IJ}\partial_i & 0 & 0 & 0 \\ 0 & 0 & -\epsilon^{0ij}\eta_{IJ} & -\epsilon^{0ij}\eta_{IJ}\partial_j \\ 0 & 0 & \epsilon^{0ji}\eta_{IJ}\partial_i & 0 \end{pmatrix} \delta^2(x-y). \tag{14}$$

This matrix is still singular. However, we have showed that there are not more constraints; the non invertibility of (14) means that the theory has a gauge symmetry. In this manner, we choose the following (gauge conditions) constraints

$$\begin{aligned} e_0^I &= 0, \\ A_0^I &= 0, \end{aligned} \tag{15}$$

which means that λ^I and θ^I are constants. Hence, we construct a new symplectic Lagrangian by adding the constraints (15) with the following ϕ_I and α_I Lagrange multipliers, obtaining

$$\mathcal{L}^{(2)} = \frac{1}{2}\epsilon^{0ij}\eta_{IJ} A_j^I \partial_0 A_i^J + \frac{1}{2}\Lambda \epsilon^{0ij}\eta_{IJ} e_j^I \partial_0 e_i^J + (\Omega_I^{(0)} + \phi_I)\dot{\lambda}^I + (\beta_I^{(0)} + \alpha_I)\dot{\theta}^I, \tag{16}$$

where we can identify the following set of symplectic variables $\xi^{(2)i}(x) = \{e^I_i, \lambda^I, A^I_i, \theta^I, \phi_I, \alpha_I\}$ and the 1-forms are given by $a^{(2)}{}_i(x) = \{\frac{1}{2}\Lambda\epsilon^{0ij}e_{Ij}, \Omega^{(0)}_I + \phi_I, \frac{1}{2}\epsilon^{0ij}A_{Ij}, \beta^{(0)}_I + \alpha_I, 0, 0\}$. By using this new set of symplectic variables, we obtain the following 24×24 symplectic matrix

$$f^{(2)}_{ij}(x,y) = \begin{pmatrix} -\Lambda\epsilon^{0ij}\eta_{IJ} & \Lambda\epsilon^{0ij}\eta_{IJ}\partial_j & 0 & 0 & 0 & 0 \\ \Lambda\epsilon^{0ji}\eta_{IJ}\partial_i & 0 & 0 & 0 & -\delta^J{}_I & 0 \\ 0 & 0 & -\epsilon^{0ij}\eta_{IJ} & \epsilon^{0ij}\eta_{IJ}\partial_j & 0 & 0 \\ 0 & 0 & \epsilon^{0ji}\eta_{IJ}\partial_i & 0 & 0 & -\delta^J{}_I \\ 0 & \delta^I{}_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^I{}_J & 0 & 0 \end{pmatrix} \delta^2(x-y), \tag{17}$$

we observe that $f^{(2)}_{ij}$ is not singular, hence, it is an invertible matrix. The inverse of the matrix (17) is given by the following 24×24 matrix

$$[f^{(2)}_{ij}(x,y)]^{-1} = \begin{pmatrix} \frac{1}{\Lambda}\epsilon_{0ij}\eta^{IJ} & 0 & 0 & 0 & \delta^I_J\partial_i & 0 \\ 0 & 0 & 0 & 0 & \delta^I_J & 0 \\ 0 & 0 & \epsilon_{0ij}\eta^{IJ} & 0 & 0 & \delta^I_J\partial_i \\ 0 & 0 & 0 & 0 & 0 & \delta^I_J \\ \delta^J_I\partial_j & -\delta^J_I & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta^J_I\partial_j & -\delta^J_I & 0 & 0 \end{pmatrix} \delta^2(x-y). \tag{18}$$

Therefore, from (18) it is possible to identify the following [FJ] generalized brackets given by

$$\{\xi^{(2)}_i(x), \xi^{(2)}_j(y)\}_{FD} = [f^{(2)}_{ij}(x,y)]^{-1}, \tag{19}$$

thus

$$\{e^I_i(x), e^J_j(y)\}_{FD} = [f^{(2)}_{11}(x,y)]^{-1} = \frac{1}{\Lambda}\epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \tag{20}$$

$$\{A^I_i(x), A^J_j(y)\}_{FD} = [f^{(2)}_{33}(x,y)]^{-1} = \epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \tag{21}$$

$$\{e^I_i(x), \phi_J(y)\}_{FD} = [f^{(2)}_{15}(x,y)]^{-1} = \delta^I_J\partial_i\delta^2(x-y), \tag{22}$$

$$\{A^I_i(x), \alpha_J(y)\}_{FD} = [f^{(2)}_{36}(x,y)]^{-1} = \delta^I_J\partial_i\delta^2(x-y), \tag{23}$$

$$\{\lambda^I(x), \phi_J(y)\}_{FD} = [f^{(2)}_{25}(x,y)]^{-1} = \delta^I_J\delta^2(x-y), \tag{24}$$

$$\{\theta^I(x), \alpha_J(y)\}_{FD} = [f^{(2)}_{46}(x,y)]^{-1} = \delta^I_J\delta^2(x-y), \tag{25}$$

we observe that the generalized [FJ] brackets are equivalent with those given in appendix A (see the Eqs. from (86) to (93)). In fact, if we consider in the Dirac brackets (86)-(93) that the second class constraints (81) are strongly identities, then the Dirac brackets will correspond to the generalized [FJ] brackets found above.

Furthermore, we will find the gauge transformations of the theory. We have seen that (12) allowed us to know if there are more constraints in the system [20]. If no new constraints arise from (12), then the zero modes of the matrix (14) will give rise to gauge transformations. In fact, let us obtain the gauge transformations of the theory, for this aim we rewrite the Lagrangian (13) in the following form

$$\mathcal{L}^{(1)} = \bar{a}_i^{(0)}\dot{\bar{\xi}}^{(0)i} + \dot{\gamma}^\alpha \Phi_\alpha^{(0)} - V^{(1)},$$

where the symplectic variables set $\bar{\xi}^{(0)i} = (e_i^I, A_i^I)$, $\gamma^\alpha = (\gamma^1 = \lambda^I, \gamma^2 = \theta^I)$, $\Phi_\alpha^{(0)} = (\Phi_1^{(0)} = \Omega_I^{(0)}, \Phi_2^{(0)} = \beta_I^{(0)})$. By using the symplectic variables we can construct a nonsingular matrix, namely $\bar{f}_{ij} = \frac{\partial \bar{a}_j}{\partial \xi^i} - \frac{\partial \bar{a}_i}{\partial \xi^j}$, constructed out with the symplectic 1-form $\bar{a}_i = (\frac{1}{2}\Lambda\epsilon^{0ij}e_{Ij}, \frac{1}{2}\epsilon^{0ij}A_{Ij})$. Hence, in terms of the $\bar{\xi}'s$ and the constraints we construct the following symplectic matrix

$$\mathfrak{f}_{ij}^{(1)}(x,y) = \begin{pmatrix} \bar{f} & (\frac{\partial \Phi^{(0)}}{\partial \xi}) \\ -(\frac{\partial \Phi^{(0)}}{\partial \xi})^T & 0 \end{pmatrix} \delta^2(x-y),$$

where

$$\left(\frac{\partial \Phi^{(0)}}{\partial \xi}\right)_{i\alpha} = \begin{pmatrix} \frac{\partial \Phi_1^{(0)}}{\partial \xi^1} & \frac{\partial \Phi_2^{(0)}}{\partial \xi^1} \\ \frac{\partial \Phi_1^{(0)}}{\partial \xi^2} & \frac{\partial \Phi_2^{(0)}}{\partial \xi^2} \\ \frac{\partial \Phi_1^{(0)}}{\partial \xi^3} & \frac{\partial \Phi_2^{(0)}}{\partial \xi^3} \\ \frac{\partial \Phi_1^{(0)}}{\partial \xi^4} & \frac{\partial \Phi_2^{(0)}}{\partial \xi^4} \end{pmatrix}.$$

It is easy to observe that the symplectic matrix $\mathfrak{f}_{ij}^{(1)}$ has zero-modes with the following structure [21]

$$v_{i\alpha} = \begin{pmatrix} (\bar{f}_{ij})(\frac{\partial \Phi_\alpha^{(0)}}{\partial \xi^j}) \\ 1_{(\alpha)} \end{pmatrix}. \quad (26)$$

In the case of gauge theories, the symplectic matrix will be non-invertible, however, the null eigenvectors of that matrix are generators of the intrinsic gauge symmetry. In fact, the gauge transformation of the theory are given by [21]

$$\begin{aligned} \delta\bar{\xi}^i &= (\bar{f}_{ij})^{-1}\frac{\partial \Phi_\alpha^{(0)}}{\partial \xi^j}\epsilon^\alpha, \\ \delta\gamma^\alpha &= \epsilon^\alpha, \end{aligned} \quad (27)$$

thus, by using (26) we can calculate the zero-modes of the matrix (14) and they are given by $(\mathrm{w}^{(1)})_1^T = (\partial_i\varepsilon^I, \varepsilon^I, 0, 0)$ and $(\mathrm{w}^{(1)})_2^T = (0, 0, \partial_i\zeta^I, \zeta^I)$. In this manner, from (27) the gauge trans-

formations are

$$\begin{aligned}\delta e_i^I &= \partial_i \epsilon^I,\\ \delta e_0^I &= \dot{\epsilon}^I,\\ \delta A_i^I &= \partial_i \zeta^I,\\ \delta A_0^I &= \dot{\zeta}^I,\end{aligned}$$

here ϵ and ζ form a set of infinitesimal parameters characterising the transformations. Therefore, we can observe that the zero-modes display the well known Abelian gauge symmetry of the model. On the other hand, in the Appendix A (see the part B) by using the Dirac method, we have fixed the gauge in oder to convert the first class constraints in second class constraints and we have calculated the corresponding Dirac's brackets among physical fields and they are given from (96) to (101), thus, in the follow section we will obtain by a different way those results by using the [FJ] framework. However, let us continue by working with the configuration space and now we will fix the following gauge

$$\begin{aligned}\partial^i e_i^I &= 0,\\ \partial^i A_i^I &= 0,\end{aligned} \tag{28}$$

by using this gauge with its corresponding Lagrange multipliers, namely, ρ_I and γ_I, the symplectic Lagrangian (13) is given by

$$\mathcal{L}^{(2)} = \frac{\epsilon^{0ij}}{2}\eta_{IJ}A_j^I\partial_0 A_i^J + \frac{\Lambda}{2}\epsilon^{0ij}\eta_{IJ}e_j^I\partial_0 e_i^J + \Omega_I^{(0)}\dot{\lambda}^I + \partial^i e_i^I\dot{\rho}_I + \beta_I^{(0)}\dot{\theta}^I + \partial^i A_i^I\dot{\gamma}_I,$$

where we can choose the symplectic variables as $\xi^{(2)i}(x) = \{e_i^I, \lambda^I, A_i^I, \theta^I, \rho_I, \gamma_I\}$ and the 1- form $a^{(2)}{}_i(x) = \{\frac{1}{2}\Lambda\epsilon^{0ij}e_{Ij}, \Omega_I^{(0)}, \frac{1}{2}\epsilon^{0ij}A_{Ij}, \beta_I^{(0)}, \partial^i e_i^I, \partial^i A_i^I\}$. Thus, by using these symplectic variables we obtain the symplectic matrix

$$f_{ij}^{(2)}(x,y) = \begin{pmatrix} -\Lambda\epsilon^{0ij}\eta_{IJ} & \Lambda\epsilon^{0ij}\eta_{IJ}\partial_j & 0 & 0 & -\delta_I^J\partial^i & 0\\ \Lambda\epsilon^{0ji}\eta_{IJ}\partial_i & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\epsilon^{0ij}\eta_{IJ} & \epsilon^{0ij}\eta_{IJ}\partial_j & 0 & -\delta_I^J\partial^i\\ 0 & 0 & \epsilon^{0ji}\eta_{IJ}\partial_i & 0 & 0 & 0\\ -\delta_J^I\partial^j & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\delta_J^I\partial^j & 0 & 0 & 0 \end{pmatrix}\delta^2(x-y),$$

we observe that this matrix is not singular, therefore, it is a invertible matrix. The inverse matrix is given by

$$[f_{ij}^{(2)}(x,y)]^{-1} = \begin{pmatrix} 0 & \epsilon_{ij}\frac{\eta^{IJ}}{\Lambda}\frac{\partial^j}{\nabla^2} & 0 & 0 & -\delta_I^J\frac{\partial_i}{\nabla^2} & 0\\ \epsilon_{ji}\frac{\eta^{IJ}}{\Lambda}\frac{\partial^i}{\nabla^2} & 0 & 0 & 0 & \delta_I^J\frac{1}{\nabla^2} & 0\\ 0 & 0 & 0 & \epsilon_{ij}\eta^{IJ}\frac{\partial^j}{\nabla^2} & 0 & -\delta_I^J\frac{\partial_i}{\nabla^2}\\ 0 & 0 & \epsilon_{ji}\eta^{IJ}\frac{\partial^i}{\nabla^2} & 0 & 0 & \delta_I^J\frac{1}{\nabla^2}\\ -\delta_J^I\frac{\partial_j}{\nabla^2} & -\delta_J^I\frac{1}{\nabla^2} & 0 & 0 & 0 & 0\\ 0 & 0 & -\delta_J^I\frac{\partial_j}{\nabla^2} & -\delta_J^I\frac{1}{\nabla^2} & 0 & 0 \end{pmatrix}\delta^2(x-y). \tag{29}$$

In this manner, we identify from (29) the following [FJ] brackets

$$
\begin{aligned}
\{e_i^I(x), \lambda^J(y)\}_{FD} &= \frac{1}{\Lambda}\epsilon_{0ij}\eta^{IJ}\frac{\partial^j}{\nabla^2}\delta^2(x-y), \\
\{e_i^I(x), \rho_J(y)\}_{FD} &= -\delta_I^J\frac{1}{\nabla^2}\delta^2(x-y), \\
\{A_i^I(x), \theta^J(y)\}_{FD} &= \epsilon_{0ij}\eta^{IJ}\frac{\partial^j}{\nabla^2}\delta^2(x-y), \\
\{A_i^I(x), \gamma_J(y)\}_{FD} &= -\delta_I^J\frac{\partial_i}{\nabla^2}\delta^2(x-y).
\end{aligned}
$$

In particular from (29) we also obtain $\{e_i^I(x), e_j^J(y)\}_{FD} = 0$ and $\{A_i^I(x), A_j^J(y)\}_{FD} = 0$, where coincide with the Dirac brackets given in (97) and (100). However, it is important to comment that by working with the gauge (28) was not possible obtain all Dirac's brackets given from (96) to (101) where the first class constraints have been converted in second class by fixing the gauge. This fact is present because in Dirac's method by fixing the gauge we have choosen a particular configuration of the fields, in particular, the configuration of the canonical momenta, and this fact do not allow in [FJ] framework to obtain the complete set of brackets because we used as symplectic variables the configuration space and the momenta are not invoked, they are labels. In order to obtain by using [FJ] all Dirac's brackets given from (96) to (101), it is necessary to work with the phase space as symplectic variables, we will clarify these points in latter subsection.
We finish this section carrying out the counting of physical degrees of freedom, for this aim we observe that in [FJ] formalism, it is not necessary to realise the classification between the constraints in first class or second class because they are at the same footing. Hence, this fact allow us to carry out the counting of physical degrees of freedom in a standar way, namely, there are 12 (e_i^I, A_i^I) canonical variables and there are 12 independent constraints ($\Omega_I^{(0)}, \beta_I^{(0)}, \partial^i e_i^I, \partial^i A_i^I$), thus, the degrees of freedom=Canonical Variables $-$ constraints$= 6 - 6 = 0$. In this manner, we conclude that the abelian exotic action lacks of physical degrees of freedom, i.e., it defines a topological field theory as expected.

A. Faddeev-Jackiw quantization introducing the phase space as symplectic variables

Now, in this section we will study the action (3) by means the [FJ] formalism introducing the phase space as symplectic variables. In this manner, from (3) we identify the momenta $(\pi_I^\alpha, p_I^\alpha)$ canonically conjugate to (A_α^I, e_α^I) given by

$$
\begin{aligned}
\pi_I^i &= \frac{1}{2}\epsilon^{0ij}A_{jI}, \\
p_I^i &= \frac{\Lambda}{2}\epsilon^{0ij}e_{jI}.
\end{aligned}
\tag{30}
$$

By using the canonical momenta into the Lagrangian (3), we obtain the following symplectic Lagrangian

$$
\mathcal{L}^{(0)} = \pi_I^i \dot{A}_i^I + p_I^i \dot{e}_i^I - V^{(0)}, \tag{31}
$$

where $V^{(0)} = -2A_0^I\partial_i\pi_I^i - 2e_0^I\partial_i p_I^i$. In this manner, we can identify the following set of symplectic variables $\xi^{(0)i}(x) = \{e_i^I, p_I^i, e_0^I, A_i^I, \pi_I^i, A_0^I\}$ and the components of the symplectic 1-form are $a^{(0)}{}_i(x) = \{p_I^i, 0, 0, \pi_I^i, 0, 0\}$. By using these symplectic variables, the symplectic matrix (5) is given by

$$f_{ij}^{(0)}(x,y) = \begin{pmatrix} 0 & -\delta^i{}_j\delta^J{}_I & 0 & 0 & 0 & 0 \\ \delta^j{}_i\delta^I{}_J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^i{}_j\delta^J{}_I & 0 \\ 0 & 0 & 0 & \delta^i{}_j\delta^I{}_J & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\delta^2(x-y). \tag{32}$$

We realise that this matrix is singular and thus the system has constraints. In order to obtain these constraints, we calculate the modes of the matrix (32) given by $(v_i^{(0)})_1^T = (0, 0, v^{e_0^I}, 0, 0, 0)$ and $(v_i^{(0)})_2^T = (0, 0, 0, 0, 0, v^{A_0^I})$, where $v^{e_0^I}$ and $v^{A_0^I}$ are arbitrary functions. In this manner, just like we performed the [FJ] analysis in last section, by using these modes we obtain the following constraints

$$\begin{aligned} \Omega_I^{(0)} &= \int d^2x (v^{(0)})_i^T(x)\frac{\delta}{\delta\xi^{(0)i}(x)}\int d^2y V^{(0)}(\xi) \\ &= \int d^2x v^{e_0^I}(x)[-2\partial_i p_I^i] \to [-2\partial_i p_I^i] = 0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} \Theta_I^{(0)} &= \int d^2x (v^{(0)})_i^T(x)\frac{\delta}{\delta\xi^{(0)i}(x)}\int d^2y V^{(0)}(\xi) \\ &= \int d^2x v^{A_0}(x)[-2\partial_i \pi_I^i] \to [-2\partial_i \pi_I^i] = 0. \end{aligned} \tag{34}$$

We can observe that these constraints are the secondary constraints given in (78) and obtained by mean Dirac's method. In order to find out more constraints, we form the following matrix [20]

$$f_{kj}\dot{\xi}^j = Z_k(\xi), \tag{35}$$

where

$$Z_k(\xi) = \begin{pmatrix} \frac{\partial V^{(0)}(\xi)}{\partial\xi^i} \\ 0 \\ 0 \end{pmatrix}, \tag{36}$$

and

$$f_{kj} = \begin{pmatrix} f_{ij}^{(0)} \\ \frac{\partial\Omega^{(0)}}{\partial\xi^i} \\ \frac{\partial\Theta^{(0)}}{\partial\xi^i} \end{pmatrix} = \begin{pmatrix} 0 & -\delta^i{}_j\delta^J{}_I & 0 & 0 & 0 & 0 \\ \delta^j{}_i\delta^I{}_J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^i{}_j\delta^J{}_I & 0 \\ 0 & 0 & 0 & \delta^j{}_i\delta^J{}_I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\partial_j\delta^J{}_I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\partial_j\delta^J{}_I & 0 \end{pmatrix}\delta(x-y). \tag{37}$$

The matrix (37) is obviously not a square matrix as expected, but it still has linearly independent modes, these modes are given by $(v^{(1)})_1^T = (2\partial_i v^{\lambda}, 0, v^{e_0^I}, 0, 0, 0, v^{\lambda}, 0)$ and $(v^{(1)})_2^T = (0, 0, 0, 2\partial_i v^{\alpha}, 0, v^{A_0^I}, 0, v^{\alpha})$. Furthermore, the contraction of the modes $(v^{(1)})_k$ with (36) will lead to more constraints, this is

$$(v^{(1)})_k^T Z_k \mid_{\Omega^{(0)}, \Theta_I^{(0)}=0} = 0, \tag{38}$$

with $k = 1, 2$. It is easy to observe that (38) corresponds to an identity, therefore, in [FJ] method there are not more constraints for the theory under study.

In order to construct a new symplectic Lagrangian containing the information obtained above, we introduce the Lagrangian multipliers λ^I and ρ^I associated to the constraints, this is

$$\mathcal{L}^{(1)} = \pi_I^i \dot{A}_i^I + p_I^i \dot{e}_i^I + \Omega_I^{(0)} \dot{\lambda}^I + \Theta_I^{(0)} \dot{\rho}^I \tag{39}$$

where the symplectic potential $V^{(1)} = V^{(0)} \mid_{\Omega_I^{(0)}, \Theta_I^{(0)}=0} = 0$ vanish. Now, from the symplectic Lagrangian (39) we can identify the following symplectic variables $\xi^{(1)i}(x) = \{e_i^I, p_I^i, \lambda^I, A_i^I, \pi_I^i, \rho^I\}$ and the symplectic 1-forms $a^{(1)}{}_i(x) = \{p_I^i, 0, \partial_i p_I^i, \pi_I^i, 0, \partial_i \pi_I^i\}$. By using these symplectic variables, we obtain the following symplectic matrix

$$f_{ij}^{(1)}(x,y) = \begin{pmatrix} 0 & -\delta^i{}_j \delta^J{}_I & 0 & 0 & 0 & 0 \\ \delta^j{}_i \delta^I{}_J & 0 & -2\delta^I{}_J \partial_i & 0 & 0 & 0 \\ 0 & -2\delta^J{}_I \partial_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta^i{}_j \delta^J{}_I & 0 \\ 0 & 0 & 0 & \delta^j{}_i \delta^I{}_J & 0 & -2\delta^I{}_J \partial_i \\ 0 & 0 & 0 & 0 & -2\delta^J{}_I \partial_j & 0 \end{pmatrix} \delta^2(x-y), \tag{40}$$

hence, $f_{ij}^{(1)}$ is a singular matrix, however, we have proved that there are not more constraints and the noninvertibility of (40) means that the theory has a gauge symmetry. In order to make invertible the matrix (40), it is necessary fix the following gauge $\partial^i e_i^I = 0$ and $\partial^i A_i^I = 0$. In this manner, we introduce new Lagrange multipliers, namely ϕ_I and θ_I, associated to the gauge fixing for constructing the following symplectic Lagrangian

$$\mathcal{L}^{(2)} = \pi_I^i \dot{A}_i^I + p_I^i \dot{e}_i^I + (2\partial_i p_I^i) \dot{\lambda}^I + (2\partial_i \pi_I^i) \dot{\rho}^I + (\partial^i e_i^I) \dot{\phi}_I + (\partial^i A_i^I) \dot{\theta}_I. \tag{41}$$

Hence, from (41) we identify the following symplectic variables $\xi^{(2)i}(x) = \{e_i^I, p_I^i, \lambda^I, A_i^I, \pi_I^i, \rho^I, \phi_I, \theta_I\}$ and the symplectic 1-form $a^{(2)}{}_i(x) = \{p_I^i, 0, 2\partial_i p_I^i, \pi_I^i, 0, 2\partial_i \pi_I^i, \partial^i e_i^I, \partial^i A_i^I\}$. In this manner, by using these symplectic variables, we

obtain the following symplectic matrix

$$
f^{(2)}_{ij}(x,y) = \begin{pmatrix}
0 & -\delta^i{}_j\delta^J{}_I & 0 & 0 & 0 & 0 & -\delta^J{}_I\partial_i & 0 \\
\delta^j{}_i\delta^I{}_J & 0 & -2\delta^I{}_J\partial_i & 0 & 0 & 0 & 0 & 0 \\
0 & -2\delta^J{}_I\partial_j & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta^i{}_j\delta^J{}_I & 0 & 0 & -\delta^J{}_I\partial_i \\
0 & 0 & 0 & \delta^j{}_i\delta^I{}_J & 0 & -2\delta^I{}_J\partial_i & 0 & 0 \\
0 & 0 & 0 & 0 & -2\delta^J{}_I\partial_j & 0 & 0 & 0 \\
-\delta^I{}_J\partial_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\delta^I{}_J\partial_j & 0 & 0 & 0 & 0
\end{pmatrix}\delta^2(x-y) \quad (42)
$$

We can observe that this matrix is not singular. Its inverse is given by

$$
[f^{(2)}_{ij}(x,y)]^{-1} =
$$

$$
\begin{pmatrix}
0 & \delta^I{}_J(\delta^j{}_i - \frac{\partial_i\partial^j}{\nabla^2}) & 0 & 0 & 0 & 0 & -\delta^I{}_J\frac{\partial_i}{\nabla^2} & 0 \\
-\delta^J{}_I(\delta^i{}_j - \frac{\partial_j\partial^i}{\nabla^2}) & 0 & -\frac{1}{2}\delta^J{}_I\frac{\partial^i}{\nabla^2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2}\delta^I{}_J\frac{\partial^j}{\nabla^2} & 0 & 0 & 0 & 0 & -\delta^I{}_J\frac{1}{2}\frac{1}{\nabla^2} & 0 \\
0 & 0 & 0 & 0 & \delta^I{}_J(\delta^j{}_i - \frac{\partial_i\partial^j}{\nabla^2}) & 0 & 0 & -\delta^I{}_J\frac{\partial_i}{\nabla^2} \\
0 & 0 & 0 & -\delta^J{}_I(\delta^i{}_j - \frac{\partial_j\partial^i}{\nabla^2}) & 0 & -\frac{1}{2}\delta^J{}_I\frac{\partial^i}{\nabla^2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}\delta^I{}_J\frac{\partial^j}{\nabla^2} & 0 & 0 & -\frac{1}{2}\delta^I{}_J\frac{1}{\nabla^2} \\
-\delta^J{}_I\frac{\partial_j}{\nabla^2} & 0 & \frac{1}{2}\delta^J{}_I\frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\delta^J{}_I\frac{\partial_j}{\nabla^2} & 0 & \frac{1}{2}\delta^J{}_I\frac{1}{\nabla^2} & 0 & 0
\end{pmatrix}\delta(x-y), \quad (43)
$$

from which, it is possible identify the generalized [FJ] brackets

$$
\{\xi^{(2)}_i(x), \xi^{(2)}_j(y)\}_{FD} = [f^{(2)}_{ij}(x,y)]^{-1}. \tag{44}
$$

Therefore we find the following brackets

$$
\{e^I_i(x), p^j_J(y)\}_{FD} = \delta^I{}_J(\delta^j{}_i - \frac{\partial_i\partial^j}{\nabla^2})\delta(x-y), \tag{45}
$$

$$
\{A^I_i(x), \pi^j_J(y)\}_{FD} = \delta^I{}_J(\delta^j{}_i - \frac{\partial_i\partial^j}{\nabla^2})\delta(x-y), \tag{46}
$$

$$
\{e^I_i(x), e^J_j(y)\}_{FD} = 0, \tag{47}
$$

$$
\{A^I_i(x), A^J_j(y)\}_{FD} = 0, \tag{48}
$$

$$
\{p^i_I(x), p^j_J(y)\}_{FD} = 0, \tag{49}
$$

$$
\{\pi^i_I(x), \pi^j_J(y)\}_{FD} = 0, \tag{50}
$$

$$
\{p^i_I(x), \lambda^J(y)\}_{FD} = -\frac{1}{2}\delta^J_I\frac{\partial^i}{\nabla^2}\delta(x-y), \tag{51}
$$

$$\{\pi_I^i(x), \rho^J(y)\}_{FD} = -\frac{1}{2}\delta_I^J \frac{\partial^i}{\nabla^2}\delta(x-y), \tag{52}$$

$$\{\lambda^I(x), \phi_J(y)\}_{FD} = -\frac{1}{2}\delta_J^I \frac{1}{\nabla^2}\delta(x-y), \tag{53}$$

$$\{\rho^I(x), \theta_J(y)\}_{FD} = \frac{1}{2}\delta_J^I \frac{1}{\nabla^2}\delta(x-y), \tag{54}$$

$$\{e_i^I(x), \phi_J(y)\}_{FD} = -\delta_J^I \frac{\partial_i}{\nabla^2}\delta(x-y), \tag{55}$$

$$\{A_i^I(x), \theta_J(y)\}_{FD} = -\delta_J^I \frac{\partial_i}{\nabla^2}\delta(x-y). \tag{56}$$

In this manner, we can observe that the generalized [FJ] brackets coincide with the Dirac ones obtained in the appendix A (see subsection B) expressed from (96) to (101). Therefore, we finish this section with some comments. We have reproduced by means of a different way the results obtained by using the Dirac method applied to the Abelian exotic action. In particular, the [FJ] formalism allowed us to obtain the constraints of the theory, the gauge transformations and we have carried out the counting of physical degrees of freedom, we have also showed that if in Dirac's framework we fix or not the gauge and we construct the Dirac's brackets, then the generalized [FJ] brackets coincide to each other. In the following section we will perform the [FJ] analysis for the non-Abelian theory, and we will reproduce by means a different and economic way the results reported in [17].

III. FADDEEV-JACKIW QUANTIZATION FOR AN EXOTIC ACTION FOR GRAVITY

Now, we will extend the results obtained in previous sections by analysing the non-Abelian action. In particular, we will reproduce the results reported in [17] where a pure Dirac's analysis was performed. In this respect, in [17] was reported the complete structure of the constraints, the Dirac brackets were constructed by eliminating only the second class constraints and also all those results were compared with the results obtained by means of the canonical covariant analysis. Hence, in this section we will obtain the results reported in [17] by means of [FJ] approach. In order to archive this aim, we have seen above that if Dirac's brackets are constructed by eliminating only the second class constraints, then in [FJ] it is necessary to work with the configuration space as symplectic variables. In fact, from (1) we can identify the following symplectic Lagrangian

$$\mathcal{L}^{(0)} = \epsilon^{0ij}\eta_{IJ}A_j^I\partial_0 A_i^J + \Lambda\epsilon^{0ij}\eta_{IJ}e_j^I\partial_0 e_J^i - V^{(0)}, \tag{57}$$

where the symplectic potential is given by $V^{(0)} = -2\epsilon^{0ij}\eta_{IJ}\left[F^J{}_{ij} + \frac{\Lambda}{2}\epsilon^J{}_{KL}e_i^K e_j^L\right]A_0{}^I - 2\Lambda\epsilon^{0ij}\eta_{IJ}D_i e_j{}^I e_0^J$. Thus, from (57) we identify the following symplectic variables $\xi^{(0)i}(x) = \{e_i^I, e_0^I, A_i^I, A_0^I\}$ and the symplectic 1-form $a^{(0)}{}_i(x) = \{\Lambda\epsilon^{0ij}e_{Ij}, 0, \epsilon^{0ij}A_{Ij}, 0\}$. In this manner, by

using these symplectic variables, we find that the symplectic matrix has the form

$$f^{(0)}_{ij}(x,y) = \begin{pmatrix} -2\Lambda\epsilon^{0ij}\eta_{IJ} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2\epsilon^{0ij}\eta_{IJ} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^2(x-y), \tag{58}$$

where we can observe that matrix is singular, this means that there are constraints. We calculate the modes of this matrix; these modes are given by $\widetilde{v}^{(0)}_k = (0, v^{e^I_0}(x), 0, 0)$ and $\widetilde{w}^{(0)}_k = (0, 0, 0, w^{A^I_0}(x))$, where $v^{A^I_0}$ and $v^{e^I_0}$ are arbitrary functions. So, just like was performed in previous sections, we will contract the null vector with the variation of the symplectic potential in order to obtain the constraints, this is

$$\begin{aligned}
\Omega^{(0)}_I &= \int d^2x (\widetilde{v}^{(0)})^T_i(x) \frac{\delta}{\delta\xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\
&= \int d^2x v^{e^I_0}(x) \Big[-2\Lambda\epsilon^{0ij}\eta_{IJ} D_i e_j{}^J \Big] \rightarrow \Big[-2\Lambda\epsilon^{0ij}\eta_{IJ} D_i e_j{}^J \Big] = 0, \\
\beta^{(0)}_I &= \int d^2x (\widetilde{w}^{(0)})^T_i(x) \frac{\delta}{\delta\xi^{(0)i}(x)} \int d^2y V^{(0)}(\xi) \\
&= \int d^2x w^{A_0}(x) \Big[-2\epsilon^{0ij}\eta_{IJ} \Big[F^J{}_{ij} + \frac{\Lambda}{2}\epsilon^J{}_{KL} e^K_i e^L_j \Big] \Big] \rightarrow \Big[-2\epsilon^{0ij}\eta_{IJ} \Big[F^J{}_{ij} + \frac{\Lambda}{2}\epsilon^J{}_{KL} e^K_i e^L_j \Big] \Big] = 0,
\end{aligned}$$

thus we identify the following constraints

$$\Omega^{(0)}_I = 2\Lambda\epsilon^{0ij}\eta_{IJ} D_i e_j{}^J = 0,$$

$$\beta^{(0)}_I = 2\epsilon^{0ij}\eta_{IJ} \left[F^J{}_{ij} + \frac{\Lambda}{2}\epsilon^J{}_{KL} e^K_i e^L_j \right] = 0,$$

these constraints are the secondary constraints found by means Dirac's method and reported in [17]. It is easy to prove that for this theory there are not more [FJ] constraints. In fact, the matrix

$$f^{(1)}_{kj}\dot{\xi}^j = Z_k(\xi), \tag{59}$$

has the following modes

$$\begin{aligned}
(v^{(1)})^T_1 &= (\partial_i v^\lambda + \epsilon^I{}_{JK} A^J_i v^\lambda + \epsilon^I{}_{JK} e^K_i v^\beta, v^{e^I_0}, 0, 0, v^\lambda, v^\beta), \\
(v^{(1)})^T_2 &= (0, 0, \partial_i v^\beta + \epsilon^I{}_{JK} A^J_i v^\beta + \Lambda\epsilon^I{}_{JK} e^K_i v^\lambda, v^{A^I_0}, v^\lambda, v^\beta),
\end{aligned} \tag{60}$$

and the contraction of these modes with Z_k yield identities, therefore, there are not more constraints. By following with the method, we will introduce all this information into the symplectic Lagrangian in order to construct a new one, thus, we introduce the Lagrangian multipliers λ^I, θ^I associated with the constraints $\Omega^{(0)}_I$ and $\beta^{(0)}_I$ respectively. In this manner, the new symplectic Lagrangian is given by

$$\mathcal{L}^{(1)} = \epsilon^{0ij}\eta_{IJ} A^I_j \partial_0 A^J_i + \Lambda\epsilon^{0ij}\eta_{IJ} e^I_j \partial_0 e^i_J + (\Omega^{(0)}_I)\dot{\lambda}^I + (\beta^{(0)}_I)\dot{\theta}^I, \tag{61}$$

where we can observe that the symplectic potential vanishes $V^{(1)} = V^{(0)} \mid_{\Omega^{(0)}_I = 0, \beta^{(0)}_I = 0} = 0$, this is an expected result, reflecting the general covariance of the theory just like it is present in General

Relativity.

Now, from (61) we identify the new set of symplectic variables $\xi^{(1)i}(x) = \{e_i^I, \lambda^I, A_i^I, \theta^I, \}$ and the symplectic 1-forms $a^{(1)}{}_i(x) = \{\Lambda\epsilon^{0ij}e_{Ij}, \Omega_I^{(0)}, \epsilon^{0ij}A_{Ij}, \beta_I^{(0)}\}$. By using the symplectic variables we can calculate the following symplectic matrix

$$f_{ij}^{(1)}(x,y) =$$

$$\begin{pmatrix} -2\Lambda\epsilon^{0ij}\eta_{IJ} & -2\Lambda\epsilon^{0ij}(\eta_{IJ}\partial_j + \epsilon_{IJK}A_j^K) & 0 & -2\Lambda\epsilon^{0ij}\epsilon_{IJK}e_j^K \\ 2\Lambda\epsilon^{0ji}(\eta_{IJ}\partial_i - \epsilon_{IJK}A_i^K) & 0 & -2\Lambda\epsilon^{0ji}\epsilon_{IJK}e_i^K & 0 \\ 0 & -2\Lambda\epsilon^{0ij}\epsilon_{IJK}e_j^K & -2\epsilon^{0ij}\eta_{IJ} & -2\epsilon^{0ij}(\eta_{IJ}\partial_j + \epsilon_{IJK}A_j^K) \\ -2\Lambda\epsilon^{0ji}\epsilon_{IJK}e_i^K & 0 & 2\epsilon^{0ji}(\eta_{IJ}\partial_i - \epsilon_{IJK}A_i^K) & 0 \end{pmatrix}\delta^2(x-y), \tag{62}$$

we can observe that $f_{ij}^{(1)}$ is singular, however, we have commented that there are not more constraints; the noninvertibility of (62) indicate that the theory has a gauge symmetry. Hence, we choose the following gauge fixing as constraints

$$\begin{aligned} A_0^I(x) &= 0, \\ e_0^I(x) &= 0, \end{aligned}$$

then we introduce the Lagrangians multipliers ϕ_I and α_I associated with the above gauge fixing for constructing a new symplectic Lagrangian

$$\mathcal{L}^{(2)} = \epsilon^{0ij}\eta_{IJ}A_j^I\partial_0 A_i^J + \Lambda\epsilon^{0ij}\eta_{IJ}e_j^I\partial_0 e_J^i + (\Omega_I^{(0)} + \phi_I)\dot{\lambda}^I + (\beta_I^{(0)} + \alpha_I)\dot{\theta}^I, \tag{63}$$

thus, we identify the following set of symplectic variables $\xi^{(2)i}(x) = \{e_i^I, \lambda^I, A_i^I, \theta^I, \phi_I, \alpha_I\}$ and the symplectic 1-forms $a^{(2)}{}_i(x) = \{\Lambda\epsilon^{0ij}e_{Ij}, \Omega_I^{(0)} + \phi_I, \epsilon^{0ij}A_{Ij}, \beta_I^{(0)} + \alpha_I, 0, 0\}$. Furthermore, by using these symplectic variables we find that the symplectic matrix is given by

$$f_{ij}^{(2)}(x,y) =$$

$$\begin{pmatrix} -2\Lambda\epsilon^{0ij}\eta_{IJ} & -2\Lambda\epsilon^{0ij}(\eta_{IJ}\partial_j + \epsilon_{IJK}A_j^K) & 0 & -2\Lambda\epsilon^{0ij}\epsilon_{IJK}e_j^K & 0 & 0 \\ 2\Lambda\epsilon^{0ji}(\eta_{IJ}\partial_i - \epsilon_{IJK}A_i^K) & 0 & -2\Lambda\epsilon^{0ji}\epsilon_{IJK}e_i^K & 0 & -\delta_I^J & 0 \\ 0 & -2\Lambda\epsilon^{0ij}\epsilon_{IJK}e_j^K & -2\epsilon^{0ij}\eta_{IJ} & -2\epsilon^{0ij}(\eta_{IJ}\partial_j + \epsilon_{IJK}A_j^K) & 0 & 0 \\ -2\Lambda\epsilon^{0ji}\epsilon_{IJK}e_i^K & 0 & 2\epsilon^{0ji}(\eta_{IJ}\partial_i - \epsilon_{IJK}A_i^K) & 0 & 0 & -\delta_I^J \\ 0 & \delta_J^I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_J^I & 0 & 0 \end{pmatrix}\delta^2(x-y), \tag{64}$$

we observe that $f_{ij}^{(2)}$ is not singular, hence, it is an invertible matrix. After a long calculation, the inverse is given by

$$[f_{ij}^{(2)}(x,y)]^{-1} =$$

$$\begin{pmatrix} \frac{1}{2\Lambda}\epsilon_{0ij}\eta^{IJ} & 0 & 0 & 0 & -(\delta_J^I\partial_i + \epsilon^I{}_{JK}A_i^K) & -\epsilon^{0ij}\epsilon^I{}_{JK}e_i^K \\ 0 & 0 & 0 & 0 & \delta_I^J & 0 \\ 0 & 0 & \epsilon_{0ij}\frac{1}{2}\eta^{IJ} & 0 & -\Lambda\epsilon^I{}_{JK}e_i^K & -(\delta_J^I\partial_i + \epsilon^I{}_{JK}A_i^K) \\ 0 & 0 & 0 & 0 & 0 & \delta_I^J \\ (\delta_I^J\partial_j - \epsilon_I{}^J{}_K A_j^K) & -\delta_J^I & 0 & -\Lambda\epsilon_I{}^J{}_K e_j^K & 0 & 0 \\ -\epsilon_I{}^J{}_K e_j^K & 0 & (\delta_I^J\partial_j - \epsilon_I{}^J{}_K A_j^K) & -\delta_J^I & 0 & 0 \end{pmatrix}\delta^2(x-y). \tag{65}$$

Therefore, from (65) it is possible to identify the following [FJ] generalized brackets given by

$$
\begin{aligned}
\{e^I_i(x), e^J_j(y)\}_{FD} &= \frac{1}{2\Lambda}\epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \\
\{A^I_i(x), A^J_j(y)\}_{FD} &= \frac{1}{2}\epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \\
\{e^I_i(x), \phi_J(y)\}_{FD} &= (\delta^I_J\partial_i - \epsilon^I{}_{JK}A^K_i)\delta^2(x-y), \\
\{A^I_i(x), \alpha_J(y)\}_{FD} &= (\delta^I_J\partial_i - \epsilon^I{}_{JK}A^K_i)\delta^2(x-y), \\
\{e^I_i(x), \alpha_J(y)\}_{FD} &= -\epsilon^I{}_{JK}e^K_i\delta^2(x-y), \\
\{A^I_i(x), \phi_J(y)\}_{FD} &= -\Lambda\epsilon^I{}_{JK}e^K_i\delta^2(x-y), \\
\{\lambda^I(x), \phi_J(y)\}_{FD} &= \delta^J_I\delta^2(x-y), \\
\{\theta^I(x), \alpha_J(y)\}_{FD} &= \delta^J_I\delta^2(x-y).
\end{aligned}
\tag{66}
$$

It is important to comment, that the generalized [FJ] brackets coincide with those obtained by means of the Dirac method reported in [17]. In fact, if we make a redefinition of the fields introducing the momenta

$$
\begin{aligned}
p^i_I &= \Lambda\epsilon^{0ij}\eta_{IJ}e^J_j, \\
\pi^i_I &= \epsilon^{0ij}\eta_{IJ}A^J_j,
\end{aligned}
\tag{67}
$$

the generalized [FJ] brackets (66) take the form

$$
\begin{aligned}
\{e^I_i(x), e^J_j(y)\}_{FD} &= \frac{1}{2\Lambda}\epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \\
\{A^I_i(x), A^J_j(y)\}_{FD} &= \frac{1}{2}\epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \\
\{e^I_i(x), p^j_J(y)\}_{FD} &= \frac{1}{2}\delta^j_i\delta^I_J\delta^2(x-y), \\
\{A^I_i(x), \pi^j_J(y)\}_{FD} &= \frac{1}{2}\delta^j_i\delta^I_J\delta^2(x-y), \\
\{p^i_J(x), p^j_J(y)\}_{FD} &= \frac{\Lambda}{2}\epsilon^{0ij}\eta_{IJ}\delta^2(x-y), \\
\{\pi^i_I(x), \pi^j_J(y)\}_{FD} &= \frac{1}{2}\epsilon^{0ij}\eta_{IJ}\delta^2(x-y),
\end{aligned}
\tag{68}
$$

which coincide with the full Dirac's brackets reported in [17].

Furthermore, with all constraints at hand, we can carryout the counting of physical degrees of freedom in the following form; there are 12 dynamical variables (e^I_i, A^I_i) and 12 constraints $(\Omega^{(0)}_I, \beta^{(0)}_I, A^I_0, e^I_0)$, therefore, the theory lacks of physical degrees of freedom.

We finish this section by calculating the gauge transformations of the theory. For this aim we calculate the modes of the matrix (62), those modes are given by

$$
\begin{aligned}
(w^{(1)})^T_1 &= (\partial_i\varepsilon^I + \epsilon^I{}_{JK}A^J_i\varepsilon^K + \epsilon^I{}_{JK}e^K_i\zeta^J, \varepsilon^I, 0, \zeta^I), \\
(w^{(1)})^T_2 &= (0, \varepsilon^I, \partial_i\zeta^I + \epsilon^I{}_{JK}A^J_i\zeta^K + \Lambda\epsilon^I{}_{JK}e^K_i\varepsilon^J, \zeta^I).
\end{aligned}
$$

In agreement with the [FJ] symplectic formalism, the zero-modes $(w^{(1)})^T_1$ and $(w^{(1)})^T_2$ are the generators of infinitesimal gauge transformations of the action (57) and are given by

$$\begin{aligned}
\delta e_i^I(x) &= D_i \varepsilon^I + \epsilon^I{}_{JK} e_i^K \zeta^J, \\
\delta e_0^I(x) &= \partial_0 \varepsilon^I, \\
\delta A_i^I(x) &= D_i \zeta^I + \Lambda \epsilon^I{}_{JK} e_i^K \varepsilon^J, \\
\delta A_0^I(x) &= \partial_0 \zeta^I.
\end{aligned}$$

In this manner, by using the [FJ] symplectic framework we have reproduced the fundamental gauge transformations corresponding to a Λ-deformed $ISO(2,1)$ Poincaré transformations reported in [17].

IV. CONCLUSIONS

In this paper a detailed Hamiltonian and [FJ] analysis for an Abelian exotic action and for a non-Abelian exotic action for gravity in three dimensions have been performed. With respect to the Abelian theory, by using the [FJ] we have found the constraints, the gauge transformation, we had carried out the counting of physical degrees of freedom and we have obtained the generalized [FJ] brackets. We could observe that if we work with the configuration space as symplectic variables, then we reproduce the results found by means the Dirac approach where Dirac's brackets are constructed by eliminating only the second class constraints. On the other hand, if in Dirac's framework we convert the first class constraints, in second class constraints by fixing the gauge and we calculate the new Dirac's brackets, then in order to reproduce those results, in [FJ] method it is necessary to work with the phase space as symplectic variables. We showed that by fixing or not the gauge, the Dirac brackets and generalised [FJ] brackets coincide to each other. It is important to remark, that if in Dirac's approach we fix the gauge and then we construct the Dirac brackets, in the [FJ] scheme we could not reproduce these results by working with the configuration space. In fact, the gauge fixing in Dirac's method implies to take a particular configuration of the fields and the momenta. In [FJ] by working with the configuration space, the momenta are labels, however, if we choose the phase space as symplectic variables and we fix the gauge in order to invert the symplectic matrix, now we are choosing a particular configuration of the fields and the momenta as well, then it is possible reproduce the results obtained in the Dirac approach by fixing the gauge.

Furthermore, in the case of a non-Abelian theory, we obtained by means the [FJ] method the complete set of constraints, the gauge transformations and we carried out the counting of physical degrees of freedom. In particular, we have reproduced by means a different and economical way the results reported in [17] where was performed a pure Dirac's method.

We finish this paper with some comments. We have seen that in [FJ] framework it is not necessary to classify the constraints in second class or first class as in Dirac's method is done, and this fact allows that the [FJ] method is more convenient to perform. In this sense, we can perform the analysis to other models describing three dimensional gravity. In fact, there is an alternative model reproducing

Einstein's equations with a cosmological constant and given by [22]

$$S[A,e] = S'[A,e] + \frac{1}{\gamma}\widetilde{S}[A,e], \tag{69}$$

where $S'[A,e]$ is the Palatini action, $\widetilde{S}[A,e]$ is the exotic action analysed in this work and γ is a kind of Barbero-Immirzi parameter. In fact, the Hamiltonian analysis of the action (69) has been reported in [22], in particular, in that work the Dirac brackets have been constructed only eliminating the second class constraints. In this respect, by using the results obtained in this work, we can develop the [FJ] analysis of (69), in particular we will report an easy way for calculating the algebra between the constraints by means [FJ] framework [23].

Finally, we would to comment that we have at hand all the necessary tools for performing the [FJ] analysis of theories with a difficult Hamiltonian structure where there are present tertiary constraints just like it is present in topologically massive gravity [24–31]. In fact, it is well-known that the canonical analysis of topologically massive gravity is not easy to perform. In the analysis there are present primary, secondary and tertiary constraints. Furthermore, the classification of those constraints in second class and first class is not an easy work, there are several complications in the computations in order to identify the constraints. In fact, in topologically massive gravity there are physical degrees of freedom, however, because of the hamiltonian analysis is difficult to carry out, there are inconsistencies in the counting of physical degrees of freedom [26]. In this respect, our work could be an important tool for studying those theories in the context of [FJ]. These ideas are in progress and will be the subject of forthcoming works.

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V. APPENDIX

A. Dirac's method for an abelian exotic action

The action that we will study in this appendix is given by

$$S[A,e] = \int \frac{1}{2}\epsilon^{\mu\nu\lambda}(A^I_\mu \partial_\nu A_{\lambda I} + \Lambda e^I_\mu \partial_\nu e_{\lambda I})dx^3, \tag{70}$$

here, A_μ is the gauge potential with $\mu = 0,1,2$ denoting the space-time components. We adopt the following conventions $\epsilon^{012} = \epsilon_{012} = 1$. The equations of motion obtained from (70) are

$$\frac{\delta S[A,e]}{\delta e^I_\mu} : \qquad \epsilon^{\mu\nu\rho}\Lambda \partial_\nu {e_\rho}^I = 0, \tag{71}$$

$$\frac{\delta S[A,e]}{\delta A_\mu I} : \qquad \epsilon^{\mu\nu\rho}\partial_\nu {A_\rho}^I = 0. \tag{72}$$

We can see that (71) implies $e_{\alpha I} = \partial_\alpha f_I$, thus $g_{\mu\nu} = \eta_{IJ}\partial_\mu f^I \partial_\nu f^J$, which corresponds to (locally) Minkowski spacetime. We shall resume the complete Hamiltonian analysis of the action (70); for this aim, we perform the 2 + 1 decomposition and introducing the canonical momenta $(\pi^\alpha_I, p^\alpha_I)$ canonically conjugate to (A^I_α, e^I_α) given by

$$\pi^\lambda_I := \frac{\partial \mathcal{L}}{\partial \dot{A}^I_\lambda} = \frac{1}{2}\epsilon^{0\lambda\gamma} A_{\gamma I}, \tag{73}$$

$$p^\lambda_I := \frac{\partial \mathcal{L}}{\partial \dot{e}^I_\lambda} = \frac{\Lambda}{2}\epsilon^{0\lambda\gamma} e_{\gamma I}, \tag{74}$$

with the following fundamental Poisson brackets among the canonical variables

$$\{A^I_\mu(x), \pi^\nu_J(y)\} = \delta^\nu_\mu \delta^I_J \delta^2(x-y), \tag{75}$$

$$\{e^I_\mu(x), p^\nu_J(y)\} = \delta^\nu_\mu \delta^I_J \delta^2(x-y), \tag{76}$$

we obtain the following primary constraints

$$\begin{aligned}
\Phi^0_I &:= \pi^0_I \approx 0, \\
\Phi^i_I &:= \pi^i_I - \frac{1}{2}\epsilon^{0ij} A_{jI} \approx 0, \\
\phi^0_I &:= p^0_I \approx 0, \\
\phi^i_I &:= p^i_I - \frac{\Lambda}{2}\epsilon^{0ij} e_{jI} \approx 0.
\end{aligned} \tag{77}$$

From consistency of the primary constraints, we obtain the following secondary constraints

$$\begin{aligned}
\psi_I &:= 2\partial_i p^i_I \approx 0, \\
\theta_I &:= 2\partial_i \pi^i_I \approx 0.
\end{aligned} \tag{78}$$

For this theory there are no, third constraints. Now, from the primary and secondary constraints, we need to identify which ones correspond to first and second class. For this aim, we need to calculate the rank and the null-vectors of the following 8×8 matrix whose entries will be the Poisson brackets between primary and secondary constraints given by

$$\begin{aligned}
\{\phi^i_I(x), \phi^j_J(y)\} &= -\Lambda \epsilon^{0ij} \eta_{IJ} \delta^2(x-y), \\
\{\phi^i_I(x), \Phi^j_J(y)\} &= 0, \\
\{\phi^i_I(x), \psi_J(y)\} &= \Lambda \epsilon^{0ij} \eta_{IJ} \partial_j \delta^2(x-y). \\
\{\Phi^i_I(x), \Phi^j_J(y)\} &= -\epsilon^{0ij} \eta_{IJ} \delta^2(x-y). \\
\{\Phi^i_I(x), \theta_J(y)\} &= \epsilon^{0ij} \eta_{IJ} \partial_j \delta^2(x-y).
\end{aligned} \tag{79}$$

This matrix has rank=4 and 4 null vectors, this mean that the theory presents a set of 4 first class constraints and 4 second class constraints. In this manner, by using the null vectors, we identify the following 4 first class constraints

$$\begin{aligned}
\gamma^1_I &= p^0_I \approx 0, \\
\gamma^2_I &= 2\partial_i p^i_I - \partial_i \phi^i_I \approx 0, \\
\gamma^3_I &= \pi^0_I \approx 0, \\
\gamma^4_I &= 2\partial_i \pi^i_I - \partial_i \Phi^i_I \approx 0,
\end{aligned} \tag{80}$$

and the rank allows us identify the following 4 second class constraints

$$\begin{aligned} \chi^1_I &= p^i_I - \frac{\Lambda}{2}\epsilon^{0ij} e_{jI} \approx 0, \\ \chi^2_I &= \pi^i_I - \frac{1}{2}\epsilon^{0ij} A_{jI} \approx . \end{aligned} \tag{81}$$

A direct calculation leads to the following non zero brackets between the first and second class constraints are

$$\begin{aligned} \{\chi^1(x), \chi^1(y)\} &= -\Lambda\epsilon^{0ij}\eta_{IJ}\delta^2(x-y), \\ \{\chi^2(x), \chi^2(y)\} &= -\epsilon^{0ij}\eta_{IJ}\delta^2(x-y), \end{aligned} \tag{82}$$

these Poisson brackets have the following matrix form

$$C^{ij} = \begin{pmatrix} -\Lambda & 0 \\ 0 & -1 \end{pmatrix} \epsilon^{0ij}\eta_{IJ}\delta^2(x-y), \tag{83}$$

and its inverse will be

$$[C^{ij}]^{-1} = \begin{pmatrix} -\frac{1}{\Lambda} & 1 \\ 0 & -1 \end{pmatrix} \epsilon_{0ij}\eta^{IJ}\delta^2(x-y). \tag{84}$$

With all these results, we can eliminate the second class constraints by introducing the Dirac brackets. Hence, the Dirac brackets among two functionals A, B expressed by

$$\{A(x), B(y)\}_D = \{A(x), B(y)\}_P - \int dudv \{A(x), \zeta^i(u)\}[C^{ij}]^{-1}(u,v)\{\zeta^j(v), B(y)\}, \tag{85}$$

where $\{A(x), B(y)\}_P$ is the usual Poisson brackets between the functionals A, B, $\zeta^i(u) = (\chi^1, \chi^2)$ are the second class constraints and $C^{ij\,-1}$ is given in (84). In this manner, by using this fact we obtain the following Dirac's brackets of the theory

$$\{e^I_i(x), e^J_j(y)\}_D = \frac{1}{\Lambda}\epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \tag{86}$$

$$\{e^I_i(x), p^j_J(y)\}_D = \frac{1}{2}\delta^j_i\delta^I_J\delta^2(x-y), \tag{87}$$

$$\{p^i_I(x), p^j_J(y)\}_D = \frac{\Lambda}{4}\epsilon^{0ij}\eta_{IJ}\delta^2(x-y), \tag{88}$$

$$\{A^I_i(x), A^J_j(y)\}_D = \epsilon_{0ij}\eta^{IJ}\delta^2(x-y), \tag{89}$$

$$\{A^I_i(x), \pi^j_J(y)\}_D = \frac{1}{2}\delta^j_i\delta^I_J\delta^2(x-y), \tag{90}$$

$$\{\pi^i_I(x), \pi^j_J(y)\}_D = \frac{1}{4}\epsilon^{0ij}\eta_{IJ}\delta^2(x-y), \tag{91}$$

$$\{e^I_0(x), p^0_J(y)\}_D = \delta^I_J\delta^2(x-y), \tag{92}$$

$$\{A^I_0(x), \pi^0_J(y)\}_D = \delta^I_J\delta^2(x-y). \tag{93}$$

Therefore, in order to quantize the theory, we consider to the second class constraints (81) as strong identities and Dirac's brackets are promoted to commutator. It is worth to comment, that the Dirac brackets given above are a particular case of the nonabelian case reported in [17].

B. By fixing the gauge

In spite of we have eliminated the second class constraints, it is necessary to remove all the gauge freedom of the theory. In order to archive this aim, we need to impose gauge conditions, using for example the temporal and Coulomb gauge

$$\begin{aligned}
\Omega_1 &= e_0^I \approx 0, \\
\Omega_2 &= \partial^i e_i^I \approx 0, \\
\Omega_1 &= A_0^I \approx 0, \\
\Omega_2 &= \partial^i A_i^I \approx 0,
\end{aligned} \tag{94}$$

thus we obtain the complete set of second class constraints

$$\begin{aligned}
\chi_I^1 &= p_I^0 \approx 0, \\
\chi_I^2 &= 2\partial_i p_I^i - \partial_i \phi_I^i \approx 0, \\
\chi_I^3 &= \pi_I^0 \approx 0, \\
\chi_I^4 &= 2\partial_i \pi_I^i - \partial_i \Phi_I^i \approx 0, \\
\chi_I^5 &= p_I^i - \frac{\Lambda}{2}\epsilon^{0ij} e_{jI} \approx 0, \\
\chi_I^6 &= \pi_I^i - \frac{1}{2}\epsilon^{0ij} A_{jI}, \\
\chi_7^I &= e_0^I \approx 0, \\
\chi_8^I &= \partial^i e_i^I, \approx 0, \\
\chi_9^I &= A_0^I \approx 0, \\
\chi_{10}^I &= \partial^i A_i^I \approx 0.
\end{aligned} \tag{95}$$

Now, in order to construct the new Dirac's brackets we need to calculate the matrix C_{ij} whose entries are given by the Poisson brackets between the second class constraints. That matrix has the following form

$$C_{ij} = \begin{pmatrix}
0 & -\Lambda\eta_{IJ} & 0 & 0 & 0 & -\delta_I^J\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda\eta_{IJ} & 0 & 0 & 0 & 0 & -\delta_I^J\partial_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\delta_I^J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \delta_J^I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\delta_I^J\nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\delta_J^I\partial_1 & -\delta_J^I\partial_2 & 0 & 0 & \delta_J^I\nabla^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_{IJ} & 0 & 0 & 0 & -\delta_I^J\partial_1 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_{IJ} & 0 & 0 & 0 & 0 & -\delta_I^J\partial_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_I^J & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_J^I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_I^J\nabla^2 \\
0 & 0 & 0 & 0 & 0 & 0 & -\delta_J^I\partial_1 & -\delta_J^I\partial_2 & 0 & 0 & \delta_J^I\nabla^2 & 0
\end{pmatrix} \delta^2(x-y),$$

and its inverse is given by

$$
[C^{ij}]^{-1} = \begin{pmatrix}
0 & \Lambda^{-1}\eta^{IJ} & 0 & 0 & -\Lambda^{-1}\eta^{IJ}\frac{\partial_2}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\Lambda^{-1}\eta^{IJ} & 0 & 0 & 0 & \Lambda^{-1}\eta^{IJ}\frac{\partial_1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_I^J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\delta^I_J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\Lambda^{-1}\eta^{IJ}\frac{\partial_2}{\nabla^2} & \Lambda^{-1}\eta^{IJ}\frac{\partial_1}{\nabla^2} & 0 & 0 & 0 & \delta_I^J\frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta^I_J\frac{1}{\nabla^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta^{IJ} & 0 & 0 & -\eta^{IJ}\frac{\partial_2}{\nabla^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\eta^{IJ} & 0 & 0 & 0 & \eta^{IJ}\frac{\partial_1}{\nabla^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_I^J & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^I_J & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\eta^{IJ}\frac{\partial_2}{\nabla^2} & \eta^{IJ}\frac{\partial_1}{\nabla^2} & 0 & 0 & 0 & \delta_I^J\frac{1}{\nabla^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta^I_J\frac{1}{\nabla^2} & 0
\end{pmatrix} \delta^2(x-y),
$$

In this manner, by using the definition of Dirac's brackets among two functionals, we obtain the following Dirac's brackets among the fields

$$\{e_i^I(x), p_J^j(y)\}_D = \delta^I{}_J(\delta^j{}_i - \frac{\partial_i\partial^j}{\nabla^2})\delta(x-y), \tag{96}$$

$$\{e_i^I(x), e_j^J(y)\}_D = 0, \tag{97}$$

$$\{p_I^i(x), p_J^j(y)\}_D = 0, \tag{98}$$

$$\{A_i^I(x), \pi_J^j(y)\}_D = \delta^I{}_J(\delta^j{}_i - \frac{\partial_i\partial^j}{\nabla^2})\delta(x-y), \tag{99}$$

$$\{A_i^I(x), A_j^J(y)\}_D = 0, \tag{100}$$

$$\{\pi_I^i(x), \pi_J^j(y)\}_D = 0. \tag{101}$$

We can observe that these brackets coincide with those calculated by means the [FJ] framework.

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